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Exact stationary photon distributions due to competition between one- and two-photon absorption and emission

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Abstract. We consider steady states of the one-mode quantized field interacting with two independent baths, each characterized by the one- and two-photon absorption and emission processes. In the absense of two-photon emission, using an exact analytical solution to the master equation for the diagonal elements of the density matrix in the Fock basis in terms of the confluent hypergeometric function, we obtain simple explicit expressions for the photon distribution function and for the factorial moments in the limiting cases of weak and strong two-photon absorption. If the two-photon absorption is strong enough, the steady state exhibits a sub-Poissonian photon statistics characterizing nonclassical behaviour, but Mandel's $\mathcal{Q}$-parameter cannot be less $-\frac{1}{3}$. However, the distribution depends essentially on the temperature of the 'one-photon bath'. For weak two-photon absorption, the stationary distribution is Gaussian, provided that the temperature of the 'one-photon' bath is high enough. For an inversely populated 'one-photon' bath, the $\mathcal{Q}$-parameter is close to $\frac{1}{2}$. In a generic case of nonzero two-photon emission probability, approximate asymptotic expressions for the factorial moments are found.

1. Introduction

In many cases, the process of quantum relaxation can be described in the framework of the master equation for the statistical operator $\widehat{\rho}$ [1–3] ($\hbar = 1$)

$$\frac{\partial\hat{\rho}}{\partial t} + \mathrm{i}[\hat{H}, \hat{\rho}] = \frac{1}{2}\sum_k D_k(2\hat{A}_k\hat{\rho}\hat{A}_k^\dagger - \hat{A}_k^\dagger\hat{A}_k\hat{\rho} - \hat{\rho}\hat{A}_k^\dagger\hat{A}_k) \tag{1}$$

where $\hat{A}_k$'s ($k = 1, 2, \ldots$) may be arbitrary linear operators, and D_k's are nonnegative constants. If the system under study is an electromagnetic field mode (or an equivalent harmonic oscillator), then Hamiltonian $\hat{H}$ and each operator $\hat{A}_k$ can be expressed in terms of the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. It is not difficult to solve equation (1), if $\hat{A}_k$ coincides with $\hat{a}$ or $\hat{a}^\dagger$ [4–7]. However, the problem becomes much more complicated, if some operators $\hat{A}_k$ include the powers $\hat{a}^n$ or $(\hat{a}^\dagger)^n$, which are responsible for the *multiphoton* processes [8]. For instance, exact time-dependent solutions for the *diagonal* matrix elements of the statistical operator in the Fock basis were found in the following cases only: $k = 1$, $\hat{A}_1 = (\hat{a}^\dagger)^2$ (two-photon emission [9–11]); $k = 1$, $\hat{A}_1 = \hat{a}^2$ (two-photon absorption [10, 12–15]); $k = 1$, $\hat{A}_1 = \hat{a}^n$ with $n \geqslant 2$ (multiphoton absorption [16, 17]); $k = 1$, $\hat{A}_1 = (\hat{a}^\dagger)^n$ with $n \geqslant 2$ (multiphoton emission

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[17]). The evolution of the off-diagonal matrix elements in the case of two-photon absorption was studied in [18], and for n-photon absorption in [17]. Other references can be found, for example in [19, 20]. In all those cases, the solutions depend on a *single* coefficient D_1. Time-dependent solutions for the diagonal elements of the statistical operator in the case of two independent coefficients, with $\hat{A}_1 = \hat{a}$ and $\hat{A}_2 = \hat{a}^2$ (competition between one- and two-photon absorption), were obtained in [21, 22]).

Other exact solutions with two nonzero coefficients were found in the stationary regime only. For m-photon absorption and m-photon emission (the systems in detailed balance), $\hat{A}_1 = \hat{a}^m$, $\hat{A}_2 = (\hat{a}^\dagger)^m[1 + \gamma(\hat{n} + 1)(\hat{n} + 2)\cdots(\hat{n} + m)]^{-1/2}$, this was done in [23]. Here $\hat{n} = \hat{a}^\dagger\hat{a}$, and the coefficient γ is responsible for the saturation effect in the multiphoton generalization of the Scully–Lamb [24] single-mode laser equation (the special case of $m = 2$ was studied in [14], and the case of $\gamma = 0$ was also considered in [25]). A scheme of obtaining exact stationary solutions of the two-photon Scully–Lamb equation with single-photon losses ($\hat{A}_1 = \hat{a}$, $\hat{A}_2 = (\hat{a}^\dagger)^2[1 + \gamma(\hat{n} + 1)(\hat{n} + 2)]^{-1/2}$) was given in [26], it was generalized to the case of m-photon emission in [27]. Different approximate solutions of the stationary multiphoton laser equations were given, for example in [28–31] (two-photon emission and single-photon absorption), [32] (m-photon emission and absorption or m-photon emission and single-photon absorption), [33] (m-photon emission and absorption plus additional k-photon absorption, including the cases $m > k$ and $m < k$).

In this paper we continue the study of the competition between one- and two-photon processes which began in [22]. Now we include, besides absorption, the emission processes. In this case, it is possible to find exact analytical expressions for the *stationary* values of the diagonal matrix elements of the statistical operator, provided that only *one-photon* emission processes are present [34, 35]. In section 2 we give the explicit solution to the problem in terms of the confluent hypergeometric function. Sections 3–5 are devoted to the detailed analysis of different interesting limiting cases, when the results can be expressed in terms of more simple functions. In section 6 we obtain approximate solutions for the factorial moments in a generic case, when both one- and two-photon emission processes are present. Section 7 is devoted to a discussion and conclusion.

2. Exact solutions

In the stationary case, equation (1) with *four* operators $\hat{A}_k$: $\hat{a}$, $\hat{a}^\dagger$, $\hat{a}^2$, and $(\hat{a}^\dagger)^2$, leads to the following set of equations for the diagonal matrix elements of the statistical operator, $p_n \equiv \langle n|\hat{\rho}|n\rangle$, $n = 0, 1, \ldots$:

$$D_2^{(a)}[(n+1)(n+2)p_{n+2} - n(n-1)p_n] - D_2^{(e)}[(n+1)(n+2)p_n - n(n-1)p_{n-2}] \\ + D_1^{(a)}[(n+1)p_{n+1} - np_n] - D_1^{(e)}[(n+1)p_n - np_{n-1}] = 0. \tag{2}$$

The coefficients $D_k^{(a)}$ and $D_k^{(e)}$ are responsible for the strength of the k-photon ($k = 1, 2$) absorption and emission processes, respectively. Introducing the generating function

$$F(z,t) = \sum_{n=0}^{\infty} p_n(t)z^n \tag{3}$$

we can replace an infinite system of difference equations (2) by a single differential equation with respect to an auxiliary variable z,

$$(1+z)(1-\rho z^2)F'' + [\nu(1-\xi z) - 4\rho z(1+z)]F' - [\nu\xi + 2\rho(1+z)]F = 0 \tag{4}$$

where the new coefficients are defined as

$$\nu \equiv D_1^{(a)}/D_2^{(a)} \qquad \xi \equiv D_1^{(e)}/D_1^{(a)} \qquad \rho \equiv D_2^{(e)}/D_2^{(a)}. \tag{5}$$

Coefficient ν is nothing but the ratio of the numbers of atoms in the 'one-photon' and 'two-photon' reservoirs multiplied by the ratio of the corresponding coupling constants. The parameters ξ and ρ characterize the temperature of each bath. Indeed, if $D_2^{(a)} = D_2^{(e)} = 0$, and $\xi < 1$, then equation (4) results in the Planck distribution,

$$F_1(z) = \frac{1-\xi}{1-z\xi} \qquad p_n^{(1)} = (1-\xi)\xi^n. \tag{6}$$

Consequently, the temperature of the 'one-photon' bath can be introduced via the relation $\xi = \exp(-\hbar\omega/\kappa T)$. For $D_1^{(a)} = D_1^{(e)} = 0$, we have [23, 25]

$$\begin{aligned} F_2(z) &= \frac{(1-\rho)(1-\gamma+z\gamma)}{1-z^2\rho} \\ p_{2n+j}^{(2)} &= (1-\rho)\rho^n(\gamma+j-1)(-1)^{j-1} \qquad j = 0, 1 \end{aligned} \tag{7}$$

where the additional parameter γ, characterizing a relative weight of the odd oscillator eigenstates, is determined by the initial conditions. Evidently, a stationary solution is stable provided that $\rho < 1$. However, no restrictions on the ratio ξ/ρ exist, i.e. the temperatures of the 'one-photon' and 'two-photon' baths may be different. Moreover, the case of an inverted 'one-photon' bath with a negative temperature, $\xi > 1$, is also admissible.

Let us first consider the special case $\rho = 0$ ('cold two-photon bath'). Then equation (4) is reduced to the Kummer equation

$$xy'' + (c-x)y' - ay = 0 \tag{8}$$

so it can be solved exactly. Since the generating function $F(z)$ cannot have a singularity at $z = -1$, we choose a regular solution to (8) in the form of the confluent hypergeometric function $\Phi(a; c; x)$, and obtain the following solution to (4) (see also [35]):

$$F(z) = \frac{\Phi(1; \nu[1+\xi]; \nu\xi[1+z])}{\Phi(1; \nu[1+\xi]; 2\nu\xi)}. \tag{9}$$

It can be expressed also in terms of the incomplete gamma function [34, 35]. The normalization factor is chosen in order to guarantee the identity

$$F(1) \equiv \sum_{n=0}^{\infty} p_n \equiv 1.$$

Knowing the generating function, we can calculate the probabilities

$$p_n = (1/n!)\partial^n F/\partial z^n|_{z=0}$$

and the factorial moments

$$\mathcal{N}_m \equiv \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)p_n = \partial^m F/\partial z^m|_{z=1}.$$

Using the relation [36]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Phi(a; c; x) = \frac{(a)_n}{(c)_n}\Phi(a+n; c+n; x)$$

where

$$(a)_n \equiv a(a+1)\cdots(a+n-1)$$

we obtain the following expressions, which hold for $0 \leqslant \xi < \infty$:

$$p_n = \frac{(\xi\nu)^n\Phi(1+n; \nu[1+\xi]+n; \nu\xi)}{(\nu[1+\xi])_n\Phi(1; \nu[1+\xi]; 2\nu\xi)} \tag{10}$$

$$\mathcal{N}_m = \frac{m!(\xi\nu)^m\Phi(1+m; \nu[1+\xi]+m; 2\nu\xi)}{(\nu[1+\xi])_m\Phi(1; \nu[1+\xi]; 2\nu\xi)}. \tag{11}$$

However, in numerical calculations it is more convenient to use the integral representation [36]

$$\Phi(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \mathrm{e}^{xu}u^{a-1}(1-u)^{c-a-1}\,\mathrm{d}u \qquad c>a>0.$$

Thus, at $\nu[1+\xi] > 1$ equations (9)–(11) can be rewritten as follows

$$F(z) = \frac{I_0(\nu[1+\xi];\nu\xi[1+z])}{I_0(\nu[1+\xi];2\nu\xi)} \tag{12}$$

$$p_n = \frac{(\xi\nu)^n I_n(\nu[1+\xi];\nu\xi)}{n!I_0(\nu[1+\xi];2\nu\xi)} \tag{13}$$

$$\mathcal{N}_m = (\xi\nu)^m \frac{I_m(\nu[1+\xi];2\nu\xi)}{I_0(\nu[1+\xi];2\nu\xi)} \tag{14}$$

where

$$I_n(c;x) = \int_0^1 \mathrm{e}^{xu}u^n(1-u)^{c-2}\,\mathrm{d}u = \frac{\mathrm{d}^n}{\mathrm{d}x^n}I_0(c;x) \qquad c>1. \tag{15}$$

Now let us investigate the limiting cases of equations (9)–(11).

3. Strong two-photon absorption

If the two-photon absorption dominates over one-photon processes, $c = \nu(1+\xi) \ll 1$, then only the ground state and the first excited level are populated with noticeable probabilities. In this case, one can simplify the function $\Phi(a;c;x)$, using its definition

$$\Phi(a;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k x^k}{(c)_k k!}$$

and the relations

$$(c)_0 = 1 \qquad (c)_k \approx c(k-1)! \qquad (n+c)_k \approx (n)_k \qquad n,k \geqslant 1.$$

Finally, we obtain (cf [35])

$$p_0 = \frac{1+2\xi}{1+3\xi} + \mathcal{O}(\nu) \qquad p_1 = \frac{\xi}{1+3\xi} + \mathcal{O}(\nu) \tag{16}$$

$$p_n \approx \frac{\nu^{n-1}\xi^n}{(n-1)!(1+3\xi)} \qquad n \geqslant 2. \tag{17}$$

For the ‘hot one-photon bath’, $\xi = 1$, we obtain $p_0 = \frac{3}{4}$ and $p_1 = \frac{1}{4}$, whereas for the ‘completely inverted bath’, $\xi \gg 1$ (but $\nu\xi \ll 1$), we have $p_0 = \frac{2}{3}$ and $p_1 = \frac{1}{3}$. The photon statistics becomes sub-Poissonian:

$$\mathcal{Q} \approx -\mathcal{N}_1 \approx -\xi/(1+3\xi) \tag{18}$$

where

$$\mathcal{Q} \equiv \mathcal{N}_2/\mathcal{N}_1 - \mathcal{N}_1 \tag{19}$$

is the known Mandel’s parameter. This a typical result for a sufficiently strong two-photon absorption process [19, 20]. The boundary between the sub- and super-Poissonian statistics is given by the equation $\mathcal{Q} = 0$, which is equivalent to the relation $\nu(1+\xi) = 1$. In such a case, we have the *exact* Poissonian statistics [35], due to the relation $\Phi(a;a;x) \equiv \exp(x)$.

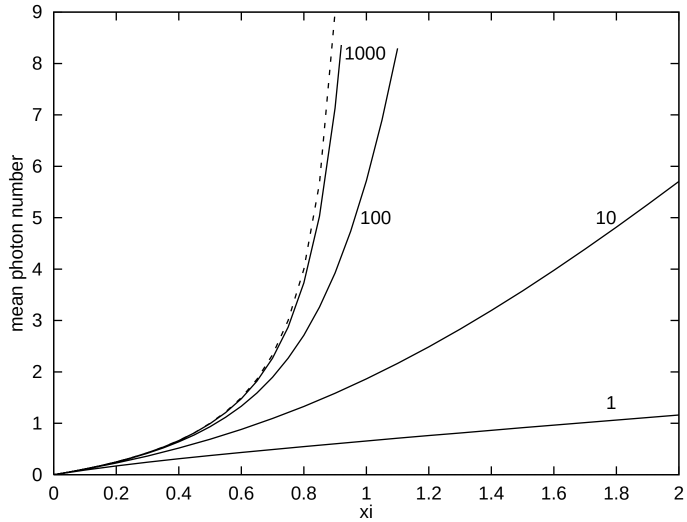

Figure 1. Mean photon number $\overline{n}$ versus $\xi \equiv D_1^{(e)}/D_1^{(a)}$. The numbers on the full curves give the values of $\nu \equiv D_1^{(a)}/D_2^{(a)}$: 1, 10, 100, 1000. The broken curve describes Planck's distribution.

4. Weak two-photon absorption

If $\nu \to \infty$ ($D_2 \to 0$), then equations (9) and (10) must yield the expressions corresponding to the Planck distribution (6). However, the limit procedure is not trivial in the high temperature case $\xi \to 1$: see figure 1. If $c \gg 1$ and $\beta < 1$, the asymptotic formula 6.13.3(18) from [36]

$$\Phi(a; c; c\beta) = (1-\beta)^{-a}\left[1 - \frac{a(a+1)\beta^2}{2c(1-\beta)^2} + \cdots\right]$$

shows that the Planck distribution can be obtained only under the conditions $\nu(1-\xi)^2 \gg 1$ and $n^2/\nu \ll 1$. To find a simple asymptotic expression, valid for *all* values of ξ, we notice that the main contribution to the integral $I_0(c; x)$ with $c \gg 1$ is given by the points in the vicinity of $u = 0$. Using the substitution

$$(1-u)^{c-2} = \exp[(c-2)\ln(1-u)] \approx \exp[-(c-2)(u+u^2/2)]$$

and extending the integration domain up to infinity, we obtain the asymptotics of $I_0(c; x)$ and $F(z)$ in terms of the error function:

$$F(z) = \frac{\varphi\left(\sqrt{\chi}(1-z\xi)\right)}{\varphi\left(\sqrt{\chi}(1-\xi)\right)} \tag{20}$$

where

$$\varphi(y) \equiv \exp(y^2)\mathrm{erfc}(y) \qquad \chi \equiv \frac{\nu}{2(1+\xi)} \tag{21}$$

and

$$\mathrm{erfc}(x) \equiv \left(2/\sqrt{\pi}\right)\int_x^\infty \exp(-t^2)\mathrm{d}t.$$

Equation (20) holds for $\nu(1+\xi) \gg 1$. Since the derivatives of function $\varphi(y)$ are expressed in terms of the parabolic cylinder function (see equation 8.2(19) from [36]),

$$\frac{\mathrm{d}^m \varphi(y)}{\mathrm{d}y^m} = \sqrt{\frac{2}{\pi}} m!(-1)^m 2^{m/2} \exp(y^2/2) D_{-m-1}\left(y\sqrt{2}\right)$$

we obtain

$$p_n = \sqrt{\frac{2}{\pi}} (2\xi^2\chi)^{n/2} \mathrm{e}^{\chi/2} \frac{D_{-n-1}\left(\sqrt{2\chi}\right)}{\varphi\left(\sqrt{\chi}(1-\xi)\right)}. \tag{22}$$

An asymptotic expression for the parabolic cylinder function of a large argument is given by equation 8.4(1) from [36]. In the case under study it can be transformed into the form

$$D_{-n-1}(x) = x^{-n-1} \mathrm{e}^{-x^2/4} \Theta_n(x) \qquad x \gg 1$$

where

$$\Theta_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(n+1)_{2k}}{k!(2x^2)^k}.$$

Replacing $(n+1)_{2k}$ by n^{2k}, we obtain a simple interpolation expression for $x \gg 1$,

$$\Theta_n(x) \approx \exp[-n^2/(2x^2)].$$

Thus we have the following asymptotics of the photon distribution at $\chi \gg 1$:

$$p_n = \frac{\xi^n \exp[-n^2/(4\chi)]}{\sqrt{\pi\chi}\varphi\left(\sqrt{\chi}(1-\xi)\right)}. \tag{23}$$

If $\sqrt{\chi}(1-\xi)) \gg 1$, this formula goes to the Planck distribution due to the asymptotics [36] $\varphi(y) \approx 1/(\sqrt{\pi}y)$. Keeping $\nu =$ constant and increasing the temperature of the first bath, we shall reach the high-temperature regime $0 \leqslant \sqrt{\chi}(1-\xi) \ll 1$. Since $\operatorname{erfc}(0) = 1$, in this case (an infinite temperature of the first bath) we obtain the *Gauss distribution* ($\chi \approx \nu/4$):

$$p_n = \frac{2}{\sqrt{\pi\nu}} \exp\left(-\frac{n^2}{\nu}\right). \tag{24}$$

The asymptotics of the mth factorial moment at $\nu(1+\xi) \gg 1$ reads

$$\mathcal{N}_m = m!\sqrt{2/\pi}(2\xi^2\chi)^{m/2} \exp[(1-\xi)^2\chi/2] D_{-n-1}\left(\sqrt{2\chi}(1-\xi)\right)/\varphi\left(\sqrt{\chi}(1-\xi)\right). \tag{25}$$

If $\sqrt{\chi}(1-\xi) \gg 1$, we obtain the Planck distribution with

$$\mathcal{N}_m = m!\left(\frac{\xi}{1-\xi}\right)^m.$$

In the opposite case, $\sqrt{\chi}(1-\xi) \ll 1$, the parabolic cylinder function can be replaced by its value at zero argument [36] $D_\alpha(0) = 2^{\alpha/2}\Gamma(\frac{1}{2})/\Gamma([1-\alpha]/2)$, so we have

$$\mathcal{N}_m = \Gamma([m+1]/2)\nu^{m/2}/\sqrt{\pi}. \tag{26}$$

This approximate formula is in a good agreement with the exact one even for $\nu \sim 1$.

5. Inverted one-photon bath

Equations (10) and (11) have no singularities at $\xi = 1$, moreover, they hold for an inversely populated 'one-photon' bath with negative temperature ($\xi > 1$). In this case it is convenient to introduce the parameters

$$\mu \equiv D_1^{(e)}/D_2^{(a)} = \nu\xi \qquad \eta \equiv 1/\xi \qquad \nu(1+\xi) \equiv \mu(1+\eta).$$

Consider equation (14) for the mth factorial moment. To calculate the function $I_m(\mu[1+\eta]; 2\mu)$ we write the integrand in equation (15) as $\exp[\varphi(u)]$, where

$$\varphi(u) = 2\mu u + [\mu(1+\eta) - 2]\ln(1-u) + m\ln u.$$

For $\mu \gg 1$, $\eta < 1$ (inverted bath), and $m \sim \mathcal{O}(1)$, the maximum of $\varphi(u)$ is achieved at $u_* \approx (1-\eta)/2$, i.e. inside the interval $(0, 1)$. Consequently, the integral can be evaluated with the aid of the steepest descent method. In the zero-order approximation one arrives at the formula $\mathcal{N}_m \approx [\mu(1-\eta)/2]^m$, meaning a Poissonian photon statistics. To obtain the first-order corrections (with respect to $1/\mu$), one has to calculate the saddle-point coordinate u_* with an accuracy of $1/\mu^2$. After some algebra we arrive at the expression

$$\mathcal{N}_m \approx \left[\frac{\mu}{2}(1-\eta)\right]^m \left(1 + \frac{m[m+1+\eta(m-3)]}{2\mu(1-\eta)^2}\right) \tag{27}$$

which shows that actually the photon statistics is *quasi-Poissonian*. When $\mu(1-\eta)^2 \to \infty$, Mandel's parameter tends to a constant value

$$\mathcal{Q}^{(\infty)}(\eta) = \frac{1+\eta}{2(1-\eta)}.$$

In the limiting case $\eta \to 0$ this result coincides with that of [34]. The dependence of the $\mathcal{Q}$-factor on parameters η and μ (calculated with the aid of the exact expressions (11) and (14)) is shown in figures 2 and 3.

The asymptotics of the photon distribution function (13) at $\mu \gg 1$ can be obtained in a similar way. In this case function $\varphi(u)$ reads

$$\varphi(u) = \mu u + [\mu(1+\eta) - 2]\ln(1-u) + n\ln u.$$

Equation (27) indicates that the distribution is concentrated in the vicinity of the number $n_* = \mu(1-\eta)/2 \gg 1$. Neglecting the term -2 in the coefficient at $\ln(1-u)$ and assuming $n = n_*(1+\varepsilon)$, $|\varepsilon| \ll 1$, we obtain in the zero approximation the same saddle point u_* as before. Performing the calculations up to the terms of the order of ε^2 (including the factor $\mu^n/n!$, which should be transformed with the aid of Stirling's formula), we arrive at the *Gauss distribution*

$$p_n = \mathcal{A}\left[\frac{2}{\pi\mu(3-\eta)}\right]^{\frac{1}{2}} \exp\left[-\frac{2[n-\mu(1-\eta)/2]^2}{\mu(3-\eta)}\right] \tag{28}$$

where the factor

$$\mathcal{A} = \left[\frac{1}{2}\operatorname{erfc}\left(-(1-\eta)\sqrt{\frac{\mu}{2(3-\eta)}}\right)\right]^{-1}$$

ensures the correct normalization of the photon distribution in the interval $0 \leqslant n < \infty$. If $\eta = 1$, equation (28) goes to (24). If $\mu(1-\eta)^2 \gg 1$, the $\mathcal{A}$-factor can be replaced by unity. Figure 4 shows a good agreement between equation (28) and exact formula (13).

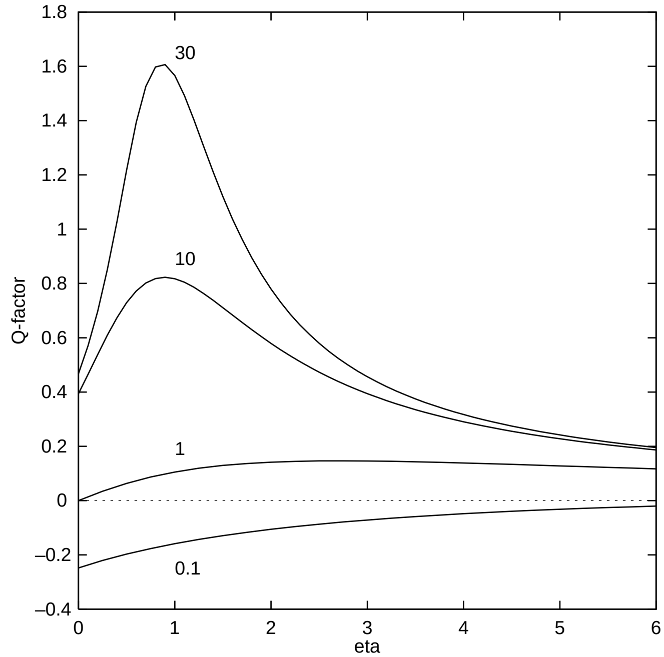

Figure 2. $\mathcal{Q}$-parameter versus $\eta \equiv D_1^{(a)}/D_1^{(e)}$. The numbers on the curves give the values of $\mu \equiv D_1^{(e)}/D_2^{(a)}$: 0.1, 1, 10, 30.

6. Approximate expressions for factorial moments in a generic case

Now let us return to the general equation (4) with $\rho \neq 0$. The generating function can be expressed in terms of the factorial moments as follows,

$$F(z) = \sum_{m=0}^{\infty} \frac{\mathcal{N}_m}{m!}(z-1)^m. \tag{29}$$

Then equation (27) shows, that at $\mu \gg 1$, $F(z)$ is close to the exponential function of the variable $x = \mu(z-1)$. Equation (4) can be rewritten in terms of this variable as

$$[2(1-\rho) + \varepsilon(1-5\rho)x - 4\varepsilon^2\rho x^2 - \varepsilon^3\rho x^3]F'' - [1 - \eta + \varepsilon(x+8\rho) + 12\varepsilon^2\rho x + 4\varepsilon^3\rho x^2] \\ \times F' - \varepsilon[1 + 4\varepsilon\rho + 2\varepsilon^2\rho x]F = 0 \tag{30}$$

where $\varepsilon = 1/\mu \ll 1$. To fix the solution, we impose the restrictions

$$F|_{x=0} = 1 \qquad F|_{x=-\infty} = 0. \tag{31}$$

The first of them is the normalization condition, while the second one simply means that $p_0 \equiv F|_{x=-\mu} \to 0$ when $\mu \to \infty$, in accordance with the exact results of the previous sections. Then in the zero approximation ($\varepsilon = 0$) we immediately obtain $F(x) = \exp(\gamma x)$, where

$$\gamma = \frac{1-\eta}{2(1-\rho)}.$$

Writing $F(x) = \exp(\gamma x)[1 + \varepsilon f(x)]$ and taking into account only linear (with respect to ε) terms in equation (30), we arrive at the inhomogeneous equation

$$2(1-\rho)f'' + (1-\eta)f' + \gamma^2(1-5\rho)x - \gamma(x+8\rho) - 1 = 0 \tag{32}$$

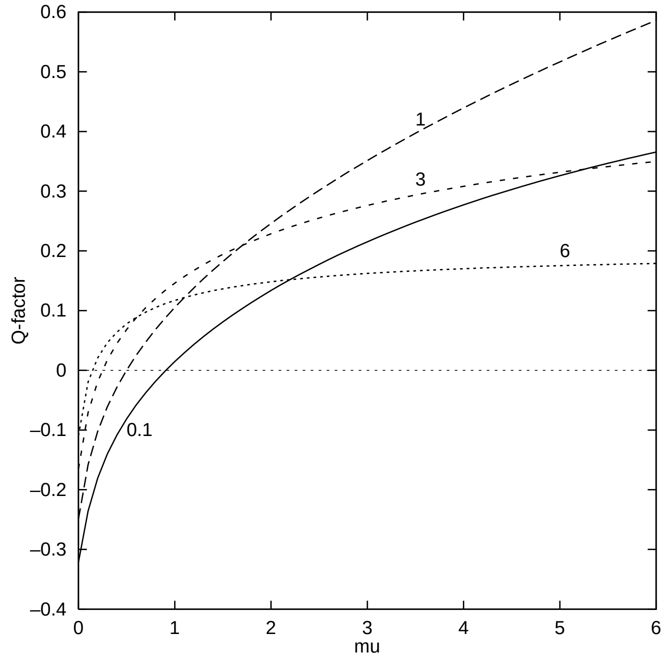

Figure 3. $\mathcal{Q}$-parameter versus $\mu \equiv D_1^{(e)}/D_2^{(a)}$. The numbers on the curves give the values of $\eta \equiv D_1^{(a)}/D_1^{(e)}$.

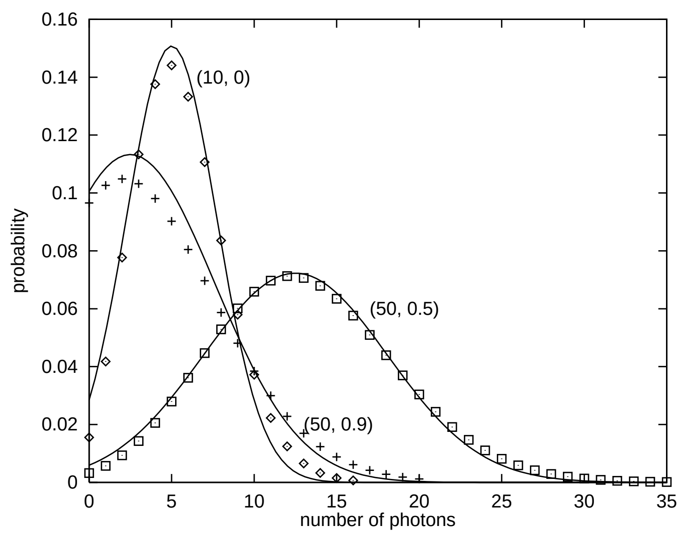

Figure 4. The photon distribution function at $\mu > 1$: points indicate exact probabilities, full curves indicate Gaussian approximation. The numbers at the curves give the corresponding pairs of values (μ, η).

whose general solution reads

$$f(x) = ax + bx^2 + A\exp(-\gamma x) + B$$
$$a = \frac{1+3\rho}{2(1-\rho)} \qquad b = \frac{1+\eta+\rho(3-5\eta)}{8(1-\rho)^2}.$$

Due to conditions (31), $A = B = 0$. Consequently, up to the corrections of the order of $1/\mu \ll 1$, we have the following asymptotic expressions for the factorial moments:

$$\mathcal{N}_1 = \frac{\mu(1-\eta)}{2(1-\rho)}\left[1 + \frac{1+3\rho}{\mu(1-\eta)} + \cdots\right] \tag{33}$$

$$\mathcal{N}_2 = \left[\frac{\mu(1-\eta)}{2(1-\rho)}\right]^2\left[1 + \frac{3+9\rho-\eta-11\rho\eta}{\mu(1-\eta)^2} + \cdots\right] \tag{34}$$

$$\frac{\mathcal{N}_m}{(\mu\gamma)^m} - 1 \approx \frac{m\,[(m+1)(1+3\rho)+(m-3)\eta-\rho\eta(1+5m)]}{2\mu(1-\eta)^2}. \tag{35}$$

Thus, the limit value of the $\mathcal{Q}$-factor equals

$$\mathcal{Q}^{(\infty)}(\eta,\rho) = \frac{1+\eta+\rho(3-5\eta)}{2(1-\eta)(1-\rho)}.$$

If $\rho = 0$, the above relations go to the formulae obtained in the preceding section.

7. Discussion

Let us formulate the main results of the paper. We have generalized the earlier studies of [34, 35], providing a detailed analysis of the exact solution to the master equation describing a stationary photon distribution arising due to a competition between one-photon emission, absorption and two-photon absorption. In particular, we have shown, that in the case of an inversely populated one-photon reservoir, the photon distribution is close to a Gaussian, and the photon statistics is quasi-Poissonian. Thus, we have rigorously confirmed the results of [28–30, 32, 33], obtained in the frameworks of different approximations. We have analysed the behaviour of various characteristics of the distribution, such as factorial moments or Mandel's $\mathcal{Q}$-factor, in all the range of variation of three parameters describing the model. Moreover, we have succeeded in finding approximate solutions for the factorial moments in a generic case, when both one- and two-photon emission processes are present.

We have shown that even weak two-photon absorption can drastically change the stationary state of the field mode. Conversely, an arbitrary small probability of one-photon emission and absorption completely changes the stationary state of the mode (in comparison with the case $D_1^{(e)} = 0$), removing the dependence of the lowest level probabilities p_0 and p_1 on the initial conditions (7) (which can be considered as some kind of degeneracy) and replacing it by the unique distribution (16), (17), which depends on the temperature of the 'one-photon bath'. For this reason, it would be interesting to find the *photon distribution* at $\nu \to 0$ and $\rho \neq 0$.

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